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Quasi-hole solutions in finite noncommutative Maxwell-Chern-Simons theory

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ABSTRACT: We study Maxwell-Chern-Simons theory in 2 noncommutative spatial dimensions and 1 temporal dimension. We consider a finite matrix model obtained by adding a linear boundary field which takes into account boundary fluctuations. The pure Chern-Simons has already been previously shown to be equivalent to the Laughlin description of the quantum Hall effect [6, 7]. With the addition of the Maxwell term, we find that there exists a rich spectrum of excitations including solitons [12] with nontrivial "magnetic flux" and quasi-holes with nontrivial "charges", which we describe in this article. The magnetic flux corresponds to vorticity in the fluid fluctuations while the charges correspond to sources of fluid fluctuations. We find that the quasi-hole solutions exhibit a gap in the spectrum of allowed charge.

KEYWORDS: Non-Commutative Geometry, Field Theories in Lower Dimensions, Chern-Simons Theories, Gauge Symmetry.

Contents

1. Introduction

In recent years noncommutative geometry has been an effervescent field of research particularly in its relation to solitons in effective descriptions of string theory and D-branes [1, 2]. However its most surprising application comes in a description of strongly correlated quantum magneto-hydrodynamics and various other quantum dynamical fluids [3]. The most intriguing application in this context is to the quantum Hall effect. Susskind [4] proposed that non-commutative Chern-Simons theory in 2+1 dimensions would be an appropriate description of the quantum Hall effect. The quantum Hall effect concerns the strongly correlated quantum dynamics of a two dimensional electron gas in a strong transverse magnetic field. The noncommutative space exists in the internal two dimensional space of the Lagrange coordinate description [5] of the electron fluid.

The continuum, classical description of the small fluctuations of a two dimensional fluid is easily seen to be a gauge theory of the group of area preserving diffeomorphisms. The gauge fields (spatial components) correspond to fluctuations of the fluid with respect to the ground state of a quiescent, undisturbed fluid. The gauge freedom of area preserving diffeomorphisms, simply corresponds to a relabeling of the elements of the fluid which are native to the Lagrange description of fluid dynamics, an evident invariance of the theory. The corresponding conservation law is equivalent to the Gauss law.

In the presence of a strong transverse magnetic field and in the low energy approximation the classical term with the lowest number of derivatives is exactly the Chern-Simons

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term. In this theory, the Gauss law in fact imposes the vortex free condition on the fluid. The vortices are frozen out of the fluid and act as sources, just as ordinary charges in electrodynamics act as sources outside of the electric and magnetic fields. Imposing the Gauss law via the introduction of a temporal gauge fields and an enhanced gauge invariance (now including time dependent gauge transformations) results in a fully non-abelian Chern-Simons gauge theory of the group of area-preserving diffeomorphisms.

Susskind's [4] key observation was that this non-abelian gauge theory appears to be a truncation to first order of the simplest noncommutative Chern-Simons gauge theory defined on two noncommutative spatial and one normal temporal dimension. Thus Susskind proposed that the true theory of the quantum Hall effect corresponds to the full noncommutative Chern-Simons gauge theory. One motivation given for this hypothesis was to reintroduce the discreteness that exists at the particulate level of the two dimensional electron gas, a discreteness which the continuum approximation erases. It remained to be seen if the phenomenology of the quantum Hall effect could be reproduced with this hypothesis. It was shown in Susskind [4] but also in more detail in [6, 7] that indeed the noncommutative gauge theoretical description of the quantum Hall effect when restricted to a finite droplet of the fluid through the introduction of a boundary and boundary degrees of freedom, was in one to one correspondence to the description afforded by the Laughlin wave functions [8]. However, the probability densities calculated in the noncommutative Chern-Simons model was only equal to that of the Laughlin wave functions in large distance limit [9]. This theory however describes the quantum Hall state via the projection to the lowest Landau level, it cannot hope to describe any transition between levels or the transition to the final state called the Hall insulator [10] for very strong external magnetic field.

In the absence of the transverse magnetic field, the lowest order term in the effective Lagrangian corresponds to the Maxwell term for the gauge field of area-preserving gauge transformations. The coefficient of the "electric" part does not have to be correlated with the coefficient of the "magnetic" part combining to give a relativistically invariant action, however this can be arranged by re-scaling the gauge field or time variable appropriately. Again the Gauss law constraint can be obtained by the incorporation of a temporal component to the gauge field and writing a gauge invariant expression for the field strength. The Maxwell term is the next order term that can be added to the pure Chern-Simons gauge theory. It renders the theory more interesting, the Gauss law constraint does not expel vortices from the theory but imposes a more dynamical constraint. We have already studied this theory in previous articles, where we found plane wave solutions for the unbounded theory [11] and soliton solutions for the theory of a finite droplet [12]. In this article we further examine the theory on a finite droplet and show the existence of quasi-hole states and rotational excitations. With this rich spectrum of excitations we expect that the theory should be able to describe transitions as a function of the parameters. Other authors [13] have looked for the quantum solution of the noncommutative Maxwell-Chern-Simon theory and found the correspondance to be to more than one Landau level. However, in order to find their solution they had to assume that a certain deformation energy of the fluid (defined in the next section) was either zero or infinity. In this article, we find classical

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quasi-hole solutions for the noncommutative Maxwell-Chern-Simon theory for arbitrary values of the deformation energy.

2. The model, equations of motion and the Hamiltonian

Susskind's [4] idea was to describe a two (spatial) dimensional fluid by a gauge field A_j so that

$$x^i = y^i + \theta\epsilon^{ij}A_j \tag{2.1}$$

where $x_i,\ i = 1,2$ are the Eulerian coordinates of the fluid and $y_i,\ i = 1,2$ are the Lagrangian (comoving) coordinates of the fluid and $\theta = 1/(2\pi\rho_0)$. Then the Lagrangian of a charged fluid in an external transverse magnetic field corresponds to the 2+1 dimensional Maxwell-Chern-Simons theory for small value of the gauge field. The continuum approximation removes the discreteness that is manifest in the physical fluid. Susskind proposed to bring this discreteness back by suggesting that the noncommutative version of this theory should describe the full theory. We studied this theory with the additional modification of boundary and boundary degrees of freedom [12]. Here we study the same action, however we add a factor κ to the analog of the magnetic field squared term which corresponds to the potential energy density of spatial deformations of the fluid. The first term, the analog of the electric field squared, corresponds to the kinetic term, the Chern-Simons term represents the interaction of the charged fluid with the external magnetic and the last term represents the boundary degrees of freedom:

$$\begin{aligned} S = \frac{\pi}{g^2\theta}\int dt(Tr\{(-2[D_0,D][D_0,D^\dagger] - \kappa[D,D^\dagger][D,D^\dagger])) \\ +2\lambda(-[D,D^\dagger]+1)D_0\} - 2\Psi^\dagger D_0\Psi). \end{aligned} \tag{2.2}$$

Here D_0 is the time covariante derivative, D and $D^\dagger$ are the holomorphic and anti-holomorphic combinations of the spatial covariant derivatives respectively and Ψ is a boundary field. The boundary field was first added by Polychronakos [6] which allowed him to find solutions of non-commutative Chern-Simons theory in terms of finite matrices. These correspond to finite droplets of the quantum Hall fluid. Specifically

$$D_\mu = \sqrt{\theta}(-i\partial_\mu + A_\mu) \tag{2.3}$$

and

$$D = \frac{D_1 + iD_2}{\sqrt{2}}, \qquad D^\dagger = \frac{D_1 - iD_2}{\sqrt{2}} \tag{2.4}$$

and the parameters λ and g^2 are related to the noncommutativity parameter θ, the electron charge e, the external magnetic field B, the density ρ_0 and the electron mass m by

$$\lambda = \frac{eB\theta^{1/2}}{m}, \qquad g^2 = \frac{(2\pi)^2\rho_0}{m}. \tag{2.5}$$

We rescale D_0 and Ψ and the parameters in the following way, in order to obtain exactly the action studied in [12]:

$$D_0 = \sqrt{\frac{\theta}{\kappa}}(-i\partial_0 + A_0), \quad \lambda = \frac{eB\theta^{1/2}}{m\sqrt{\kappa}}, \quad g^2 = \frac{(2\pi)^2\rho_0}{m\kappa}, \quad \Psi \to \frac{\Psi}{\sqrt[4]{\kappa}} \tag{2.6}$$

Defining

$$\Xi = \frac{\pi}{g^2\theta} \tag{2.7}$$

we obtain the action

$$S = \Xi \int dt(Tr\{(-2[D_0, D][D_0, D^\dagger] - [D, D^\dagger][D, D^\dagger])) \\ +2\lambda(-[D, D^\dagger] + 1)D_0\} - 2\Psi^\dagger D_0 \Psi). \tag{2.8}$$

By varying with respect to $\Psi^\dagger$, D_0, and $D^\dagger$ we get, respectively, the boundary equation

$$i\dot{\Psi} = A_0 \Psi, \tag{2.9}$$

the Gauss law

$$[D, [D_0, D^\dagger]] + [D^\dagger, [D_0, D]] + \lambda([D, D^\dagger] - 1) + \Psi\Psi^\dagger = 0, \tag{2.10}$$

and the Ampère law

$$[D_0, [D_0, D]] + [D, [D, D^\dagger]] = \lambda[D_0, D]. \tag{2.11}$$

The Hamiltonian is, as in [12], given by

$$H = \Xi\ Tr(-2[D_0, D^\dagger][D_0, D] + [D, D^\dagger][D, D^\dagger]). \tag{2.12}$$

3. Rotational excitations

Our first solution corresponds to rotational excitations on top of any given solution. Our procedure can be applied to the soliton solutions found for example in [12] and to the solutions that we find in this article. We put

$$D = D' + \frac{R}{\sqrt{2\theta}} e^{iwt},\ D_0 = D_0',\ \Psi = \Psi' \tag{3.1}$$

where the primed variables correspond to any known solution to the equations of motion with R (proportional to the identity) and w (real) simply constant. The Gauss law involves commutators of the D or $D^\dagger$ with each other or with their commutator with D_0. The direct addition of complex constants as in the equation (3.1) or those that result from the commutators involving D_0, simply vanish open taking the further commutators with D or $D^\dagger$, hence the Gauss law is satisfied. The equation (2.9) is also obviously satisfied. Replacing (3.1) into the Ampère law yields a solution if $w = \frac{\lambda\sqrt{\kappa}}{\sqrt{\theta}} = \frac{eB}{m}$ the familiar cyclotron frequency. We calculate the corresponding change of energy:

$$\Delta H = -2\Xi Tr\Big([D_0, \frac{R^*}{\sqrt{2\theta}} e^{-iwt}][D_0, D'] + [D_0, D'^\dagger][D_0, \frac{R}{\sqrt{2\theta}} e^{iwt}] \\ +[D_0, \frac{R^*}{\sqrt{2\theta}} e^{-iwt}][D_0, \frac{R}{\sqrt{2\theta}} e^{iwt}]\Big) \tag{3.2}$$

which yields

$$\Delta H = -2\Xi Tr\left(-\lambda \frac{R^*}{\sqrt{2\theta}} e^{-iwt}[D_0, D'] + \lambda \frac{R}{\sqrt{2\theta}} e^{iwt}[D_0, D'^\dagger] - \lambda^2 \frac{R^* R}{2\theta}\right). \tag{3.3}$$

Since the commutator $[D_0, D]$ is off-diagonal in our solution and the solution in [12] the trace vanishes giving

$$\Delta H = \frac{2\Xi\lambda^2|R|^2 N}{2\theta}. \tag{3.4}$$

Expressing this in terms of physical constants gives

$$\Delta H = \frac{e^2 B^2 |R|^2 N}{2m} \tag{3.5}$$

which is exactly the energy of N rotating electrons with a amplitude R at the frequency w in a magnetic field.

This solution adds a term to the covariant derivative

$$D_1 = D_1' + \frac{|R|}{\sqrt{\theta}}\cos(wt+\varphi), D_2 = D_2' + \frac{|R|}{\sqrt{\theta}}\sin(wt+\varphi) \tag{3.6}$$

where φ is the phase of R. Using the Susskind correspondence between the fluid coordinates and the gauge field (2.1), we see that the solution (3.1) corresponds to rotating about the original solution as

$$y_1 = y_1' - |R|\sin(wt+\varphi),\ \ y_2 = y_2' + |R|\cos(wt+\varphi) \tag{3.7}$$

where y_1' and y_2' are for operator coordinates of original solution. These rotational excitations clearly also exist in the analogous theory in the infinite plane treated in [11].

4. Quasi-hole solutions

Quasi-hole solutions appears as a modification of the solutions found in [12]. We will take an ansatz similar to [12], hence D is represented by an $N \times N$ matrix which satisfies certain boundary conditions, however we will take a periodic D as in [6]. The D is an operator similar in structure to the annihilation operator of an ordinary Heisenberg algebra, hence it generally relates a state $|n+1\rangle$ to a state $|n\rangle$. For the finite matrix representations used here, periodicity means that the final state $|0\rangle$ is related to the state $|N-1\rangle$ since n ranges over the N values $0, 1, \ldots N-1$.

$$D = \sum_{n=0}^{N-2} \sqrt{G(n) + q e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}}} \mid n\rangle\langle n+1 \mid + \sqrt{q} e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}} \mid N-1\rangle\langle 0 \mid \tag{4.1}$$

If $q = 0$, this ansatz is equivalent to the one in [12]. $G(n)$ is to be determined from the equations of motion, we solve these eventually, perturbatively and numerically.

Solutions with this ansatz correspond to quasi-hole solutions because they bound the lowest eigenvalue of the radius away from zero in a q dependent manner. From (2.1), the fluctuation of the radius is proportional to the square of the gauge field

$$A_1^2 + A_2^2 = \left(\frac{D_1}{\sqrt{\theta}} + i\partial_1\right)^2 + \left(\frac{D_2}{\sqrt{\theta}} + i\partial_2\right)^2. \tag{4.2}$$

Noncommutative geometry in a finite space is defined by the commutator of the coordinates $[x_1, x_2] = i\theta(1 - N \mid N-1\rangle\langle N-1 \mid)$ which imply the commutator for the derivate $[\partial_1, \partial_2] = \frac{i}{\theta}(-1 + N \mid N-1\rangle\langle N-1 \mid)$. Thus we can put

$$\begin{aligned} \partial_1 &= \frac{i}{\sqrt{2\theta}}(d + d^\dagger), \\ \partial_2 &= \frac{1}{\sqrt{2\theta}}(d - d^\dagger) \end{aligned} \tag{4.3}$$

where

$$d = \sum_{n=0}^{N-2} \sqrt{n+1} \mid n\rangle\langle n+1 \mid . \tag{4.4}$$

Then we obtain

$$R^2 \propto \{D - d, D^\dagger - d^\dagger\} \tag{4.5}$$

where $\{D^\dagger, d\}$ is the anti-commutator. Then the radius becomes

$$\begin{aligned} R^2 \propto & \left(G(0) + 2q - 1 - 2\sqrt{G(0)+q}\cos\left(\frac{w(0)\sqrt{\kappa}t}{\sqrt{\theta}}\right)\right) \mid 0\rangle\langle 0 \mid \\ & + \sum_{n=1}^{N-2}\left(\underline{G(n)} + 2q - 2n - 1 - \underline{2\sqrt{G(n)+q}\sqrt{n+1}\cos\left(\frac{w(n)\sqrt{\kappa}t}{\sqrt{\theta}}\right)}\right) \mid n\rangle\langle n \mid \\ & + \left(G(N-2) + 2q - N + 1 - 2\sqrt{G(N-2)+q}\sqrt{N-1}\cos\left(\frac{w(N-2)\sqrt{\kappa}t}{\sqrt{\theta}}\right)\right) \times \\ & \times \mid N-1\rangle\langle N-1 \mid \end{aligned} \tag{4.6}$$

a diagonal expression in the states where $\underline{A(n)} = A(n) + A(n-1)$. We see that for large q we have correspondingly large eigenvalues for R^2. The smallest eigenvalue is not directly equal to q, hence our solution corresponds to the fluid pushed away from the origin.

Returning to the solution of the equations of motion, we will consider the gauge $A_0 = 0$ but we let D depend on time. This choice is different from that taken in [12], however in that case, our choice is simply gauge equivalent. With the periodic ansatz of equation (4.1), this is not the case. Indeed, our choice give us an additional degree of freedom which we can identify as ρ. Thus $D_0 = -i\sqrt{\theta}\partial_0$, and we can calculate the different terms in the equations of motion:

$$\begin{aligned} [D_0, D] &= \sum_{n=0}^{N-2} w(n)\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}} \mid n\rangle\langle n+1 \mid + \rho\sqrt{q}e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}} \mid N-1\rangle\langle 0 \mid \\ [D_0, [D_0, D]] &= \sum_{n=0}^{N-2}(w(n))^2\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}} \mid n\rangle\langle n+1 \mid + \rho^2\sqrt{q}e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}} \mid N-1\rangle\langle 0 \mid \end{aligned} \tag{4.7}$$

The commutator $[D, D^\dagger]$ is given by replacing from equation (4.1)

$$\begin{aligned}[D, D^\dagger] &= \sum_{n=0}^{N-2}\sum_{m=0}^{N-2}\sqrt{G(n)+q}\sqrt{G(m)+q}e^{\frac{i\sqrt{\kappa}(w(n)-w(m))t}{\sqrt{\theta}}}[\mid n\rangle\langle n+1\mid, \mid m+1\rangle\langle m\mid] \\ &\quad +q[\mid N-1\rangle\langle 0\mid, \mid 0\rangle\langle N-1\mid] \\ &= \sum_{n=0}^{N-2}(G(n)+q)\mid n\rangle\langle n\mid - \sum_{n=1}^{N-1}(G(n-1)+q)\mid n\rangle\langle n\mid + q(\mid N-1\rangle\langle N-1\mid - \mid 0\rangle\langle 0\mid) \\ &= \sum_{n=0}^{N-1}\overline{G(n-1)}\mid n\rangle\langle n\mid \end{aligned} \tag{4.8}$$

with the notation $\overline{A(n)} \equiv A(n+1)-A(n)$ and we define $G(-1)\equiv 0$ and $G(N-1)\equiv 0$. The commutator $[D^\dagger, [D_0, D]]$ can be computed in the same way as $[D, D^\dagger]$. Defining $w(-1)\equiv \rho$ and $w(N-1)\equiv \rho$ then

$$[D^\dagger, [D_0, D]] = -\sum_{n=0}^{N-1}\overline{(G(n-1)+q)w(n-1)}\mid n\rangle\langle n\mid . \tag{4.9}$$

As this result is hermitian it is also equal to $[D, [D_0, D^\dagger]]$. The final term appearing in the equations of motion is calculated as

$$\begin{aligned}[D, [D, D^\dagger]] &= \sum_{n=0}^{N-2}\sum_{m=0}^{N-1}\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}}\overline{G(m-1)}[\mid n\rangle\langle n+1\mid, \mid m\rangle\langle m\mid] \qquad (4.10) \\ &\quad +\sqrt{q}e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}}\sum_{m=0}^{N-1}\overline{G(m-1)}[\mid N-1\rangle\langle 0\mid, \mid m\rangle\langle m\mid] \\ &= \sum_{n=0}^{N-2}\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}}\overline{\overline{G(n-1)}}\mid n\rangle\langle n+1\mid + \\ &\quad +\sqrt{q}e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}}\left(\overline{G(-1)}-\overline{G(N-2)}\right)\mid N-1\rangle\langle 0\mid \end{aligned}$$

with the notation $\overline{\overline{G(n-1)}} \equiv \nabla^2 G(n) = G(n+1)-2G(n)+G(n-1)$ where $\nabla^2 G(n)$ is discrete Laplacian (which is facilitated with the further notational convenience $G(N)\equiv G(0)$ and $\langle N\mid \equiv \langle 0\mid$). Then

$$[D, [D, D^\dagger]] = \sum_{n=0}^{N-1}\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}}\nabla^2 G(n)\mid n\rangle\langle n+1\mid . \tag{4.11}$$

The solution for Ψ is simply a general static vector since $A_0 = 0$ in (2.9):

$$\Psi = \sum_{n=0}^{N-1}\lambda_n \mid n\rangle. \tag{4.12}$$

We see, as in [12], that only $\Psi\Psi^\dagger$ contributes off diagonal terms in the Gauss law (2.10). To eliminate such terms we must take $\Psi = \lambda_M \mid M\rangle$. Contrary to [12], different choices of

M are all gauge equivalent (i.e. a permutation) since all choices of M are equivalent in our periodic ansatz (up to the name of the variable). Taking the trace of the Gauss law then yields $\lambda_M = \sqrt{N\lambda}$. We choose without loss of generality that $M = N-1$. Then the Gauss law yields (for $n = 0, \ldots N-1$)

$$-2\overline{(G(n-1)+q)w(n-1)} + \lambda\left(\overline{G(n-1)} - 1\right) + \lambda N\delta_{N-1,n} = 0. \tag{4.13}$$

We will solve this equation by induction as in [12]. We will show that for $n = [0, N-2]$

$$w(n) = -\lambda\frac{n+1-\frac{2q\rho}{\lambda} - G(n)}{2(G(n)+q)}. \tag{4.14}$$

This formula is true for $n = N-2$, as is verified by considering the Gauss law (4.13) for $n = N-1$. Then assuming the form (4.14) for a general value of n we can prove that it is valid for $n \to n-1$. Thus by the principle of induction the formula is valid for all n. However we have to check/impose that the $n = 0$ equation of (4.13) is respected. This is indeed the case if we take (4.14). Taking the Ampère law (2.11) and removing an overall factor $\sqrt{G(n)+q}e^{\frac{iw(n)\sqrt{\kappa}t}{\sqrt{\theta}}}$ we obtain, for $n = [0, N-1]$.

$$(w(n))^2 + \nabla^2 G(n) = \lambda w(n). \tag{4.15}$$

Here we have N equations, in $N-1$ values of $G(n)$ and also in the two variables q and ρ. Hence we have N equations and $N+1$ parameters, thus we have one free parameter. Generically, there will be a family of solutions. The Hamiltonian, from equation 2.12, for this ansatz, using equations 4.7 and 4.8 is

$$H = \Xi(\sum_{n=0}^{N-1} 2(w(n))^2(G(n)+q) + \left(\overline{G(n-1)})^2\right). \tag{4.16}$$

4.1 Perturbative analysis of the quasi-hole solution

4.1.1 Large quasi-hole solution

The equation (4.15) is non-linear and thus difficult to solve. So we will look the solution for $q \gg 1$. We define

$$\rho = \rho_0 + \frac{\rho_1}{q},\ G(n) = G_0(n) + \frac{G_1(n)}{q} \tag{4.17}$$

Then if we expand w to first order in $1/q$, we obtain:

$$w(n) \simeq \rho_0 - \frac{1}{2q}(\lambda(n+1-G_0(n)) - 2\rho_1 + 2\rho_0 G_0(n)) \tag{4.18}$$

$$(w(n))^2 \simeq \rho_0^2 - \frac{\rho_0}{q}(\lambda(n+1-G_0(n)) - 2\rho_1 + 2\rho_0 G_0(n)) \tag{4.19}$$

Working to zero order in $1/q$, equation 4.15 gives (for $n = [0, N-2]$)

$$\rho_0^2 + \nabla^2 G_0(n) = \lambda\rho_0. \tag{4.20}$$

The discrete Laplacian solved as in the continuous case by

$$G_0(n) = \alpha_0 n^2 + \beta_0 n + \delta_0, \;\; \alpha_0 = \frac{\lambda\rho_0 - \rho_0^2}{2}. \tag{4.21}$$

We have to impose the two boundary conditions $G_0(-1) = G_0(N-1) = 0$. These give

$$\alpha_0 - \beta_0 + \delta_0 = 0, \;\; \alpha_0(N-1)^2 + \beta_0(N-1) + \delta_0 = 0 \tag{4.22}$$

which are solved by

$$\beta_0 = \frac{1}{2}(N-2)(\rho_0^2 - \lambda\rho_0), \;\; \delta_0 = \frac{1}{2}(N-1)(\rho_0^2 - \lambda\rho_0). \tag{4.23}$$

We still have one final condition left (from the $n = N-1$ of equation 4.15) which gives

$$\rho_0^2 + \alpha_0(N-2)^2 + \beta_0(N-2) + 2\delta_0 = \lambda\rho_0 \tag{4.24}$$

with solutions

$$\rho_0 = 0 \qquad \text{or} \qquad \rho_0 = \lambda. \tag{4.25}$$

Either of these solutions give us $G_0(n) = 0$.

With the order zero solutions we can continue to solve the equation (4.15) in first order in $1/q$

$$-\rho_0(\lambda(n+1) - 2\rho_1) + \nabla^2 G_1(n) = \frac{-\lambda}{2}(\lambda(n+1) - 2\rho_1). \tag{4.26}$$

This is solved by

$$\begin{aligned} G_1(n) &= \alpha_1 n^3 + \beta_1 n^2 + \delta_1 n + \gamma_1, \\ \alpha_1 &= \frac{\lambda}{12}(2\rho_0 - \lambda), \\ \beta_1 &= \frac{1}{4}(2\rho_0 - \lambda)(\lambda - 2\rho_1). \end{aligned} \tag{4.27}$$

Again we have our boundary conditions $G_1(-1) = G_1(N-1) = 0$ which imply

$$\begin{aligned} \delta_1 &= \frac{1}{12}(N^2\lambda - 3\lambda - 6N\rho_1 + 12\rho_1)(\lambda - 2\rho_0), \\ \gamma_1 &= -\frac{1}{12}(N-1)(\lambda N - 6\rho_1 + \lambda)(2\rho_0 - \lambda). \end{aligned} \tag{4.28}$$

Finally the condition for $n = N-1$ of (4.15) gives

$$\rho_1 = \frac{1}{4}(N-1)\lambda. \tag{4.29}$$

Thus

$$\delta_1 = -\frac{\lambda}{24}(N^2 - 9N + 12)(\lambda - 2\rho_0) \tag{4.30}$$

$$\gamma_1 = \frac{\lambda}{24}(N-1)(N-5)(2\rho_0 - \lambda) \tag{4.31}$$

$$\beta_1 = \frac{\lambda}{8}(3-N)(2\rho_0 - \lambda) \tag{4.32}$$

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From the Hamiltonian (4.16), for the $\rho_0 = 0$ solution, it can be shown with a little calculation that a non-zero contribution arises only at order $1/q$. The variables $(\rho, G(n), w(n))$ must be expanded to order $1/q^2$ to consistently extract this contribution, because the Hamiltonian contains an explicit factor of q. Indeed, as we will see below, for the $\rho_0 = 0$ solution, the second order expansion of these variables does not in fact give a non-zero contribution, however they do contribute to the energy for the $\rho_0 = \lambda$ solution. Then we find the solution to second order:

$$G_2(n) = \alpha_2 n^5 + \beta_2 n^4 + \delta_2 n^3 + \gamma_2 n^2 + \xi_2 n + \epsilon_2, \tag{4.33}$$

$$\begin{aligned}
\alpha_2 &= \frac{\alpha}{20}, \\
\beta_2 &= \frac{\beta}{12}, \\
\delta_2 &= \frac{-\alpha + 2\delta}{12} \\
\xi_2 &= -\frac{\alpha N^4}{20} + \frac{N^3}{12}(3\alpha - \beta) + \frac{N^2}{12}(-5\alpha + 4\beta - 2\delta) \\
&\qquad + \frac{N}{12}(3\alpha - 5\beta + 6\delta - 6\gamma) + \frac{1}{6}(\beta - 3\delta + 6\gamma)
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\gamma_2 &= \frac{-\beta + 6\gamma}{12}, \\
\epsilon_2 &= \xi_2 - \frac{\alpha - 5\delta + 15\gamma}{30},
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\alpha &= \frac{\lambda^2 \alpha_1}{2}, \\
\beta &= \frac{\lambda^2}{4}(2\beta_1 - 1), \\
\delta &= \frac{\lambda}{2}(2\rho_1 + \lambda\delta_1 - \lambda) \\
\gamma &= \lambda\rho_1 - \rho_1^2 + \lambda\rho_2 - \frac{\lambda^2}{4} - 2\rho_2\rho_0 + \frac{\lambda^2\gamma_1}{2}
\end{aligned} \tag{4.36}$$

$$\rho_2 = \frac{-(N^2 - 1)(\lambda^3 + 2(2\rho_0 - \lambda))}{96}. \tag{4.37}$$

We have implicitly assumed that $q \gg \lambda$, $q \gg \rho_i$ and $q \gg G_i(n)$ for every i (or more precisely that these variables are of order of q^0). However, we find that the $G_i(n)$ are polynomial in n where $n = 0, \ldots, N$. Thus if N becomes large, $G_i(n)$ would also become large. Therefore the condition $q \gg 1$ is not adequate for the perturbative expansion to converge. We see that $G_1(n)$ and $G_2(n)$ are respectively third and fifth degree polynomials. Thus our perturbation series would not be valid if $N^3 \gg q$ or $N^5 \gg q^2$. If we replace

$$G(n) = \sum_{i=0}^{\infty} \frac{G_i(n)}{q^i}. \tag{4.38}$$

in equation (4.15) and expand to order $1/q^i$, then we obtain a recurrence equation relating the discrete Laplacian of $G_i(n)$ to $G_j(n)$ with $j < i$. Assuming that the solution for the

$G_j(n)$ with $j < i$ is a polynomial in n, the solution for $G_i(n)$ is a polynomial for degree two higher. Then knowing that $G_1(n)$ is a polynomial of order 3, by induction we see that $G_i(n)$ is a polynomial of the degree $2i+1$. Thus we see that the true perturbative parameter is actually is n^2/q. Since $n = 0, \ldots, N$, we take the strongest condition, $N^2/q \ll 1$. This condition might actually not be necessary, since we in fact impose the boundary condition that $G(N-1) = 0$, and indeed our numerical analysis agrees with the perturbative analysis even for $N^2/q \sim N$.

The Hamiltonian is made up of the kinetic and potential energy. The potential energy actually contributes at order $1/q^2$, which we will neglect. The kinetic energy contains terms of order q. These are in principle the dominant terms for large q. They have a constant energy density, that is the energy associated to each state$\mid n\rangle\langle n\mid$. In total the trace gives

$$H_{kin.}(q) = 2\Xi N \rho_0^2 q. \tag{4.39}$$

At order zero, the kinetic energy density is linear (in n), but the total after the trace is zero:

$$H_{kin.}(q^0) = 2\Xi Tr\left(-\lambda\rho_0 \sum_{n=0}^{N-1}\left(n - \frac{2\rho_1}{\lambda}\right) \mid n\rangle\langle n \mid\right) = 0. \tag{4.40}$$

For the order $1/q$ contribution we need $w(n)^2$ to order $1/q^2$.

$$(w(n))^2 \simeq \rho_0^2 - \frac{\lambda\rho_0}{q}\left(n + 1 - \frac{2\rho_1}{\lambda}\right) + \frac{1}{q^2}\left(\frac{\lambda^2}{4}\left(n + 1 - \frac{2\rho_1}{\lambda}\right)^2 + 2\rho_0\left(\rho_2 - \frac{\lambda G_1(n)}{2}\right)\right). \tag{4.41}$$

The kinetic energy then is

$$H_{kin.}(1/q) = \frac{2\Xi}{q} Tr\left(\sum_{n=0}^{N-1}\left(\frac{\lambda^2}{4}(n - \frac{2\rho_1}{\lambda})^2 + 2\rho_0\rho_2\right) \mid n\rangle\langle n \mid\right) \tag{4.42}$$

which gives

$$H_{kin.}(1/q) = \frac{2\Xi}{q}\left(\frac{\lambda^2 N}{48}(N^2 - 1) + 2\rho_0\rho_2 N\right). \tag{4.43}$$

Hence the energy up to the $1/q$ is

$$H \simeq 2\Xi N \rho_0^2 q + \frac{2\Xi}{q}\left(\frac{\lambda^2 N}{48}(N^2 - 1) + 2\rho_0\rho_2 N\right). \tag{4.44}$$

The two solutions (for ρ_0) behave quite differently for $\rho_0 = \lambda$ the energy diverges as q becomes large while for $\rho_0 = 0$ it vanishes.

$$H_{\rho_0=\lambda} \simeq 2\Xi N \lambda^2 q + \frac{2\Xi}{q}\left(\frac{\lambda^2 N}{48}(N^2 - 1) + 2\lambda\rho_2 N\right) \tag{4.45}$$

$$H_{\rho_0=0} \simeq \frac{\Xi\lambda^2 N(N^2 - 1)}{24q} \tag{4.46}$$

These solutions correspond to an annulus of large radius and small (relatively) thickness that oscillate in time. We can see for the operator of the radius squared (which is diagonal), from equation (4.6), the dominant part of the coefficient (hence eigenvalue of R^2) is proportional to $2q$ for every state, in the large q limit. The next dominant term is proportional to the square root of q and is an oscillatory term (the frequency for $\rho_0 = \lambda$ is of order q^0 and it is weakly n dependent, and for $\rho_0 = 0$ it is of order q^{-1}). The next important term is proportional to $2n+1$. As $q \gg \sqrt{q} \gg 1$ the radius is very large and oscillates with an amplitude that is relatively much smaller than the radius. Thus from afar the droplet looks like an thin annulus that undergoes a nontrivial oscillation.

4.1.2 Small quasi-hole solution

The small quasi-hole solution, for ($q \ll 1$), can be obtained by a perturbation on the solution found in [12]. Take D at first order perturbation to have the form

$$D = D' + D_p, \;\; D_0 = D'_0 + D_{0p}, \;\; \Psi = \Psi' + \Psi_p, \tag{4.47}$$

where the primed variables are solutions found in [12] and the variables subscripted p are the perturbations. To first order the Gauss law gives

$$\begin{aligned} 0 = {} & [D_p,[D'_0,D'^{\dagger}]] + [D',[D_{0p},D'^{\dagger}]] + [D',[D'_0,D^{\dagger}_p]] + [D^{\dagger}_p,[D'_0,D']] \\ & +[D'^{\dagger},[D_{0p},D']] + [D'^{\dagger},[D'_0,D_p]] + \lambda([D_p,D'^{\dagger}] + [D',D^{\dagger}_p]) + \Psi_p\Psi'^{\dagger} + \Psi'\Psi^{\dagger}_p \end{aligned} \tag{4.48}$$

while the Ampère law gives

$$\begin{aligned} & [D_{0p},[D'_0,D']] + [D'_0,[D_{0p},D']] + [D'_0,[D'_0,D_p]] + [D_p,[D',D'^{\dagger}]] \\ & \qquad +[D',[D_p,D'^{\dagger}]] + [D',[D',D^{\dagger}_p]] = \lambda[D_{0p},D'] + \lambda[D'_0,D_p] \end{aligned} \tag{4.49}$$

and the constraint on Ψ gives

$$D_{0p}\Psi' + D'_0\Psi_p = 0. \tag{4.50}$$

We assume the perturbation takes the form

$$D_p = \sqrt{q} e^{\frac{i\rho\sqrt{\kappa}t}{\sqrt{\theta}}} \mid N-1\rangle\langle 0 \mid, \;\; D_{0p} = 0, \;\; \Psi_p = 0. \tag{4.51}$$

The ansatz in [12] has the form (gauge equivalent to)

$$D' = \sum_{n=0}^{N-2} \sqrt{G(n)} e^{iw(n)t} \mid n\rangle\langle n+1 \mid, \;\; A_0 = 0 \tag{4.52}$$

where $G(n)$ and $w(n)$ are explicitly calculated (numerically) in [12]. Then the Gauss law is automatically satisfied using

$$\left[\mid N-1\rangle\langle 0 \mid, \sum_{n=0}^{N-2} f(n) \mid n+1\rangle\langle n \mid\right] = \left[\mid 0\rangle\langle N-1 \mid, \sum_{n=0}^{N-2} f(n) \mid n\rangle\langle n+1 \mid\right] = 0. \tag{4.53}$$

The Ampère law becomes

$$[D'_0,[D'_0,D_p]] + [D_p,[D',D'^{\dagger}]] = \lambda[D'_0,D_p] \tag{4.54}$$

with $[D', D'^{\dagger}]$ given by 4.8. This implies the same condition as before (removing an overall factor D_p) (4.15) however only for $n = N - 1$

$$\rho^2 D_p + (G(0) + G(N-2))D_p = \lambda \rho D_p. \tag{4.55}$$

For other values of n the condition is automatically satisfied. Then ρ is given by

$$\rho = \frac{\lambda \pm \sqrt{\lambda^2 - 4(G(0) + G(N-2))}}{2}. \tag{4.56}$$

As this expression involves a square root, we see that there is no solution for small λ, as we will confirm below, in the numerical analysis. This is a gap in the spectrum of allowed q for small λ. The change in energy is given by

$$\Delta H = -2\Xi Tr([D'_0, D_p^{\dagger}][D'_0, D'] + [D'_0, D'^{\dagger}][D'_0, D_p] + [D'_0, D_p^{\dagger}][D'_0, D_p]) \tag{4.57}$$

which gives

$$\Delta H = -2\Xi Tr(-\rho D_p^{\dagger}[D'_0, D'] + \rho[D'_0, D'^{\dagger}]D_p - \rho^2 D_p^{\dagger} D_p) \tag{4.58}$$

and using that the two first terms vanish, yields

$$\Delta H = 2\Xi \rho^2 q. \tag{4.59}$$

4.2 Numerical analysis of the quasi-hole solution

In this section, we will find numerical solutions of equation (4.15). As our solutions are built upon the solutions found in [12], we first give the numerical analysis of the equations considered there. We take this opportunity to correct certain errors that have appeared in [12]. Equation (5.6) in [12] is incorrect, the kinetic energy is not symmetric about $M = N/2$. The correct equation is:

$$T = 2\Xi\left(\left\{\sum_{n=0}^{N-2} 2g^2 u_n - u_n \nabla^2 u_n\right\} + g^2(N^2 - N - 2NM)\right) \tag{4.60}$$

Correspondingly, figure (5) in [12] is also not correct. In addition, there seems to be an inconsistency between figure (5) and figure (7) in [12], the kinetic energy appears to be greater than the total energy. We give corrected figures, figure 1 and figure 2 here (not for the same values of the parameters). These are representative solutions for the system of equations studied in [12]. Our further numerical analysis concerns solutions built upon these. We ultimately use the Newton method [14] to solve the difference equations. Finite difference equations on a finite set of variables/parameters and boundary conditions, after iterating recursively and removing dependent variables, simply become a system of extremely complicated algebraic equations in a much reduced set of variables. In our case, we reduce the system to two equations in three variables, which we take to be q, ρ and $G(N-2)$. We solve this resultant system by the Newton method. We can constrain our search a little by noting that $q + G(n) \geq 0$ since it appears under a square root in all the

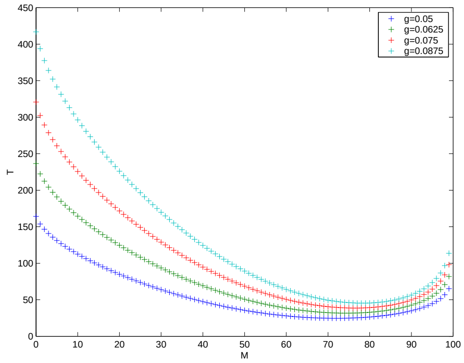

Figure 1: Kinetic energy as a function of M for N=100 for (corrected) solution in [12]

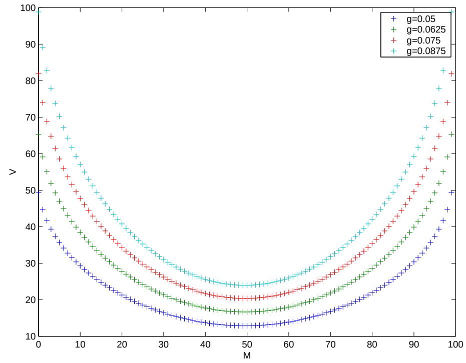

Figure 2: Potential energy as a function of M for N=100 for solution in [12]

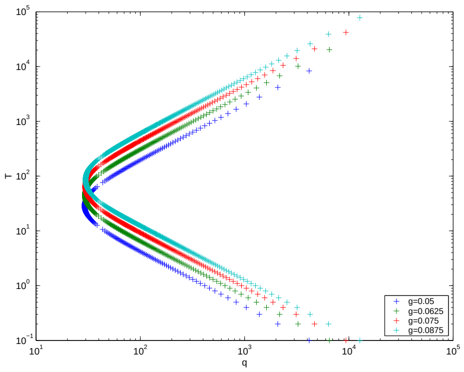

Figure 3: Kinetc energy as a function of q for $N = 100$ and for 4 values of $g = \lambda/2$ in units of Ξ.

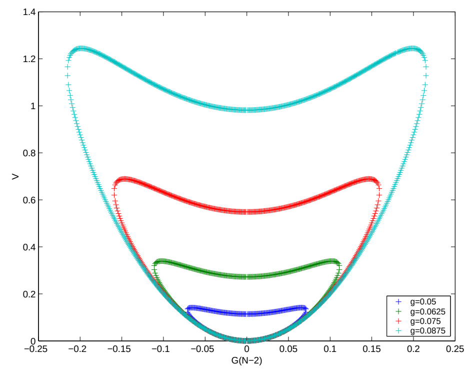

Figure 4: Potential energy as a function of $G(N-2)$ for $N = 100$ and $g = \lambda/2$ in unit of Ξ.

expressions for D. For the case $n = N - 1$, if we add $2q$ to the equation (4.15) then we get the constraint

$$\rho \in \left[\frac{\lambda - \sqrt{\lambda^2 + 8q}}{2}, \frac{\lambda + \sqrt{\lambda^2 + 8q}}{2} \right] \tag{4.61}$$

Numerically, we easily find the two solutions that we have determined analytically, for $q \gg 1$. We see in figure (3) that the kinetic energy either diverges as q or vanishes as $1/q$ as $q \to \infty$, depending on which branch we consider, exactly as we have seen in the section (4.1.1). For the potential energy, we give a graph as a function of the value of $G(N-2)$. The preceding two branches of solutions, are found in figure (4), on the bottom part of the curve, symmetrically on either side of the point $G(N-2) = 0$. This dependence on $G(N-2)$ can be inferred from the analytic solution in section (4.1.1). We can see in figure (3) that there exists a region of transition between the two branches of solutions. This transition region seems to imply a lower limit on the permitted value of q for given λ. The corresponding transition region in figure (4) is the upper part of the solution between the two peaks. Further numerical analysis varying λ seems to indicate that the transition

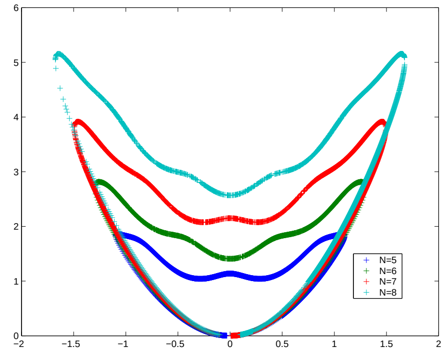

Figure 5: Potential energy as a function of $G(N-2)$ for $g = 1$

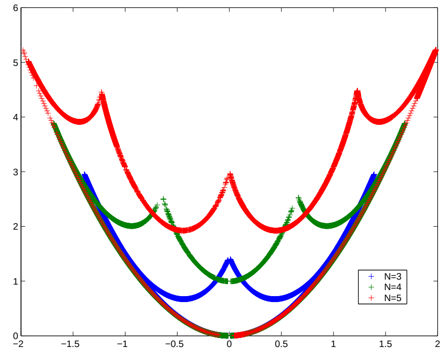

Figure 6: Potential energy as a function of $G(N-2)$ for $g = 1.5$

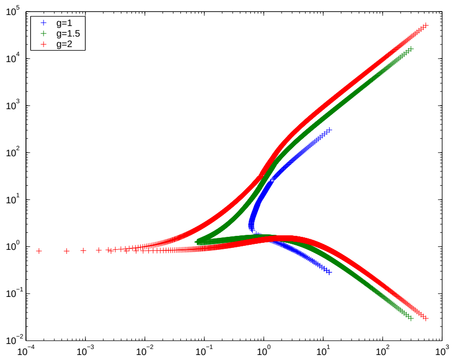

Figure 7: Kinetic energy as a function of q for $N = 3$

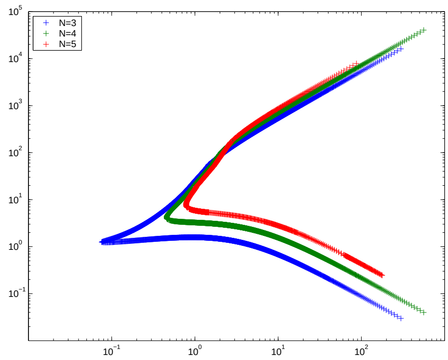

Figure 8: Kinetic energy as a function of q for $g = 1.5$

region actually extends all the way to $q = 0$ for sufficiently large λ. The $q = 0$ limit corresponds to the case studied in [12], where it was found that there are N solutions for the noncommutative droplet. We find that we recover this multiplicity in the numerical analysis. Specifically the curve of the potential energy, figure (5) and figure (6) obtains N peaks in the transition region, each of which extends down to the solution found in [12] for different choices of Ψ, as seen in figures (7) and (8). (One has to be careful in this limit for $\Psi \not\sim |N-1\rangle$. One recovers the solution of [12] for $(G(N-2-M)+q) \to 0$ and not simply $q \to 0$.) The total energy is always dominated by the kinetic term and the range of the total energy is from 0 to ∞, however the interesting structure in the potential energy implies that its contribution to the specific heat could be most important. Comparing our solution to the one found in [12], figures (1) and (2), we find that the potential energy, figure (4), of our solution is at least one order of magnitude smaller than that found in [12] for small λ and will approach the potential energy of the solutions in [12] for large λ (and small q). The kinetic energy is, however, quite different. In our solution, the kinetic energy can take any value since q is a free parameter, while the solution in [12] it is constrained to a discrete set of N values. For the values of λ used in our figures, the kinetic energy of the

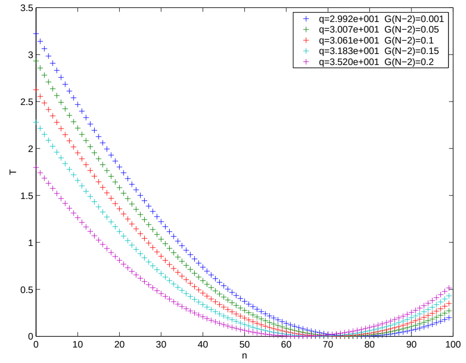

Figure 9: Kinetic density energy for $N = 100$ and $g = 0.0875$

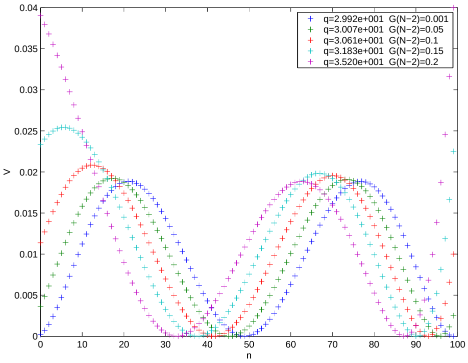

Figure 10: Potential density energy for $N = 100$ and $g = 0.0875$

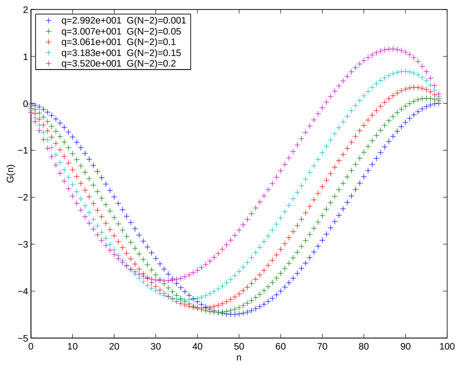

Figure 11: $G(n)$ for $N = 100$ and $g = 0.0875$

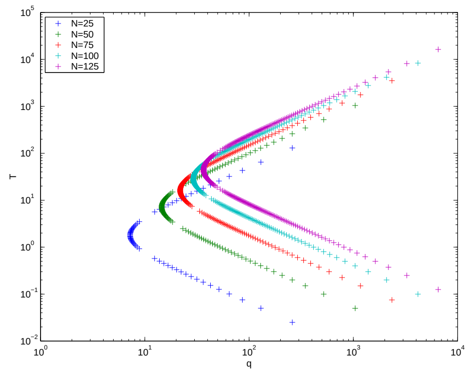

Figure 12: Kinetic energy as a function of q for $g = 0.05$

solution in [12] is comparable to the kinetic energy of our solution in the transition region.

The detail of the transition region is found in the figures (9), (10) and (11) which describes the right half of the transition region ($G(N-2) > 0$) in the figure (4). The potential energy density is seen to translate as we vary $G(N-2)$. This translation continues for $G(N-2) < 0$ symmetrically with respect to the behavior for $G(N-2) > 0$. The potential energy is found to concentrate in the bulk of the droplet, away from the boundaries in the middle of the transition regions. During the transition the kinetic density evolves from a linear density ($\rho_0 = \lambda$) as in equation (4.40) to a quadratic density ($\rho_0 = 0$) as in equation (4.42).

Away from the transition region (i.e. for $q \gg 1$, we do not give a figure since this region can be computed analytically), the form of the potential energy density and the $G(n)$'s does not change markedly, apart from their overall size. The potential energy density is concentrated around the boundary.

Finally, in the figure (12), we show the solutions for various values of N. We see that the minimum value of q seems to be growing linearly with N. Thus, assuming that the trend continues, as $N \to \infty$ then $q_{min.} \to \infty$, thus the large q-hole solutions would not

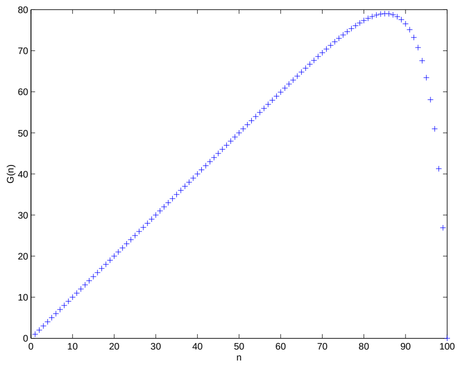

Figure 13: $G(n)$ for $M = N - 1$ with $N = 100$

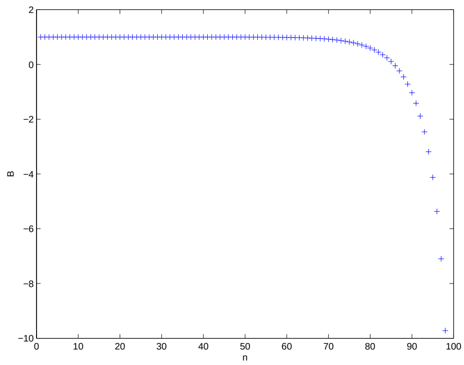

Figure 14: Vorticity ("magnetic" field B) for $M = N - 1$ with $N = 100$ (figure cut-off at $B > -10$)

occur in the infinite plane, at least for small λ.

5. Ground state

The solutions that we have found do not easily allow us to identify the ground state. For the infinite case, the ground state is given by the solution in terms of simple annihilation and creation operators, [4, 11],

$$a, \; a^\dagger, \; [a, a^\dagger] = 1. \tag{5.1}$$

This solution corresponds to a static, quiescent fluid. However the value of $[D, D^\dagger]$ is a non-zero constant, which should be considered as the zero point of energy. The solutions in the finite droplet approach this state arbitrarily closely in the limit $N \to \infty$ and for large λ. We show below in figures (13) and (14), the plot of $G(n)$ for $M = N - 1$ and with $\lambda = 2$ and the corresponding vorticity of the fluid. If we subtract out the constant background due to the noncommutativity it is evident that the fluid is almost everywhere quiescent with only net vorticity imposed near the boundary. We take these states as the ground states. We also show the corresponding kinetic and potential energy in figures (15) and (16). The kinetic energy T is concentrated at the boundary, as is the potential energy V if we subtract off the constant zero point energy.

The quasi-hole solutions that we have found are deformations of these ground states. Comparing to the case of a quasi-hole with small q, we see from equations (4.58) and (4.59) that the energy density perturbation is solely in the kinetic energy. It is also localized at the origin as is evident from the form of the perturbation equation (16). The change in the energy is linear in q. The solutions with large q are not perturbative and should be rightly considered as solitons. The question of the stability of our identified ground states against creation of solitons with large q (the branch with decreasing enenrgy, see figure (3)) is beyond the scope of this paper.

The rotational excitations that we have found presumably give the Landau levels, upon quantization. These are extremely widely separated in energy for large external magnetic

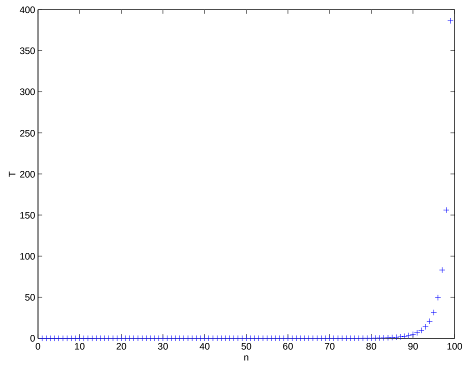

Figure 15: Kinetic energy density for $M = N - 1$ with $N = 100$

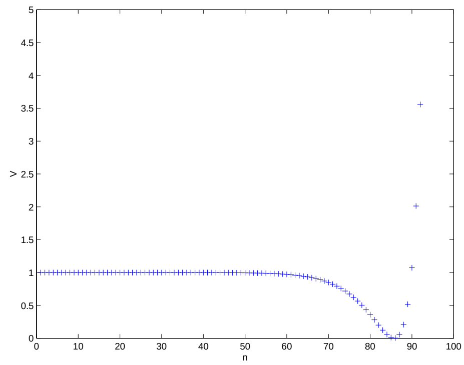

Figure 16: Potential energy density for $M = N - 1$ with $N = 100$ (figure cut-off at $V < 5$)

field and hence will decouple from small excitations. The ground state will correspond to the lowest Landau level. The degeneracy of a given Landau level is obtained from the total number of particles available and the physical space in which these particles are constrained. The Landau levels will presumably play an important role in describing the transitions between different Hall plateaux.

6. Conclusion

In this article we have found a rich spectrum of the excitations of the noncommutative droplet with an action based on the Chern-simons term and an additional Maxwell term. The existence of the Maxwell term seems to have profound implications on the spectrum of fundamental oscillations. We have found two kind of solutions, the rotational excitation and the quasi-hole solution.

The rotational excitations will exist in the finite or infinite case above any solution of the equations of motion. The rotational frequency is exactly the cyclotron frequency for the electrons in the model. The rotational excitation correspond to the energy of N electrons moving in the magnetic field when the amplitude of the rotational excitation is fixed to the radius of the cyclotron motion. With the quantization of these oscillations we should recover the familiar Landau levels.

The quasi-holes solutions seem to exist only for the finite droplet. The large quasi-holes correspond to a thin annulus that undergoes nontrivial oscillatory vibration. They give a continuous band of solutions built above each soliton solution of the field equations. They also show a gap in their spectrum as a function of the "charge" q for small enough λ.

We expect that this spectrum of excitations will give rise to a complex phenomenolgy which will allow us to describe transitions in two dimensional magnetohydrodynamics, even the quantum Hall system.

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Acknowledgments

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